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Vortons in two component Bose-Einstein condensates

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Abstract: Ever since Witten's paper on superconducting cosmic strings, there has been an interest in strings with a non-trivial core structure. In particular, it is believed that such strings can form loops, known as vortons, which are stabilized against shrinking by current and charge trapped on the string world-sheet. In this paper, we show that analogous non-trivial vortices and stable vortex loops exist in non-relativistic two-component atomic Bose-Einstein condensates. In contrast with well-studied superfluid ^{4}He, where similar vortex rings can be stable due to Magnus force only if they move, the vortex rings in two-component BECs can be stable even if they are at rest. We speculate that such vortons may have been already observed in the laboratory. Thus, it may be possible to test predictions regarding vortons in cosmology and astrophysics by doing controlled experiments in two-component BECs.

Keywords: Solitons Monopoles and Instantons, Spontaneous Symmetry Breaking, Global Symmetries, Cosmology of Theories beyond the SM.

Contents

1. Introduction

With the recent controlled creation of vortices [1, 2] in trapped atomic Bose-Einstein condensates (BECs), the study of structure and dynamics of topological defects in weakly interacting superfluids has become an active subject of experimental and theoretical research. Similar topological structures in form of vortex lines (strings) have been observed in superfluid ^{3}He and ^{4}He, conventional superconductors and high T_c superconductors. In most studied and well understood cases such as the ^{4}He system and the conventional superconductors, the order parameter can be represented by a single complex scalar field, which obtains a non-zero vacuum expectation value, spontaneously breaking a U(1) symmetry. The structure of the vortices in these cases is quite simple: the order parameter must vanish in the center of the vortex core, restoring the U(1) symmetry there, see e.g. [3]. If one makes a ring out of such a vortex, it would quickly shrink and decay due to the large string tension. The well-known quantized vortex rings, which have been produced and detected [4] in superfluid ^{4}He, were stable due to their motion that gives rise to a Magnus force equilibrating the string tension.

The subject of this work is the analysis of a less trivial and much more interesting situation when the vortex rings, dubbed vortons, can be stabilized even when they do not move. As we discuss below, such a stability cannot occur in a system described by a single scalar field; rather such a stability may occur in more complex systems, such as the two-component BECs, which are considered in this paper. The vorton stabilization mechanism is related to another effect, which is interesting in its own right: the condensation of a second component inside the vortex core.

We should note at this point that the theoretical feasibility for such a phenomenon to occur, was first discussed by E. Witten who suggested a toy model leading to existence of superconducting cosmic strings [5]. The idea of construction is simple: consider a two component system (such as a two component BEC), which is described by two complex scalar fields (ψ, ϕ) with an approximate U(2) symmetry. If the U(2) symmetry between fields ψ and ϕ is explicitly broken down to $U(1) \times U(1)$, the ψ condensation might be energetically more favorable than ϕ condensation, and the ground state will be given by $\langle\psi\rangle \neq 0$ and $\langle\phi\rangle = 0$. As is well known, such a system allows for existence of ψ vortices characterized by the phase of ψ field varying by an integer multiple of 2π as one traverses a contour around the vortex core. What Witten actually has demonstrated is that if the approximate U(2) symmetry is broken only weakly, the ϕ field may condense inside the core of the ψ string, "spontaneously breaking" the corresponding U(1) symmetry in the core. In the case when the ϕ field is electrically charged, such a string can support a persistent electric current along the string core, hence the term "superconducting cosmic strings". As will be discussed below, the question of existence and magnitude of ϕ field condensation in the vortex core depends solely on how strongly the U(2) symmetry is broken.

One should note here that the above idea that the core of the string could be in a different phase without restoration of symmetry was motivated by the grand unified theories with a typical scale of 10^{15} GeV. However, remarkably, this phenomenon was also observed experimentally and (or) predicted to occur theoretically in a variety of condensed matter systems such as superfluid ^{3}He (see [6, 7] for review) and high-T_c superconductors [8]–[11]. This idea was also recently studied in a number of high-energy and nuclear physics systems such as QCD at high baryon density [12, 13] and superconducting matter in neutron stars [15]. As we demonstrate in this paper, two component BECs continue the above long list of systems in which vortex core condensation might occur.

An important consequence of the phenomenon of core-condensation was discussed previously in the context of cosmic strings. Within Witten's original toy model, it has been shown [16]–[22] that condensation inside the core provides a way to stabilize a string loop against shrinking, see [23] for review. Indeed, a string loop is a topologically trivial object that normally shrinks due to string tension. However, if one allows for the ϕ field condensed in the core of a ψ vortex to carry a winding number along the string as well as a Noether's charge associated with its U(1) symmetry, then the conservation of both (the charge and the winding number) prevents the loop from shrinking. In fact, when both Noether's charge and winding number are present, the ϕ field carries a non-zero angular momentum perpendicular to the plane of the loop, which stabilizes the configuration. The resulting stable vortex loops, known as vortons, are called semi-topological defects since they are stabilized in part by topology and in part by energetics. One should comment here that this stability is purely mechanical in origin and to a leading approximation is independent of whether the relevant symmetry is local or global.

Up to this point, most discussion of vortons has been confined to relativistic physics (such as cosmic strings and high-density QCD), or to physics where the relevant low-energy effective lagrangian has a relativistic form (such as the SO(5) model of high T_c superconductivity). In the present work we wish to investigate whether counterparts of semi-topological defects described above exist in non-relativistic field theory, such as the one describing a mixture of two ultra cold Bose condensed gases. In this paper we discuss the structure of BECs with two internal levels. This is equivalent to a spin-1/2 fluid: the order parameter has U(2) rotational properties. We demonstrate that strings with condensation in the core do exist if the U(2) symmetry is slightly broken, and we calculate explicitly under which conditions this occurs. We then go on to construct vortons in non-relativistic systems and demonstrate their stability under certain conditions.

The application to the field of Bose gases seems particularly interesting in the wake of recent theoretical and experimental achievements in this area. On the experimental side, the long sought goal of obtaining a weakly interacting Bose gas is now achieved in ultra-cold alkali metal gases such as Rb, Na and Li [25, 26, 27]. Moreover, experimentally, it is now possible [28, 29] to have a system with two hyperfine states of ^{87}Rb condensed in the same trap; this is precisely the kind of system we consider in the present work. Also, as we already mentioned, in the past few years experiments have confirmed that rotational vortices do indeed form in the ultra-cold Bose gases [1, 2, 30]. Actually, the original experiment that produced a quantum vortex in ^{87}Rb [1] for technical reasons made use of a second condensate in the core of the vortex. Thus, we speculate that the phenomenon of core condensation described in this paper might have been already observed in BECs. However, since in the original experiment the second field component was artificially induced in the core, one cannot immediately identify the phenomenon of natural core condensation that we are discussing with the experimentally observed result. Moreover, we note that vortex rings, which are the subject of the present work, may have also been observed [31]. However, an additional analysis is required before one can convincingly argue that the observed rings are precisely vortons, which are stable due to the condensation in the core rather than due to some other factors such as nonzero velocity of the ring or due to the influence of the trap potential.

On the theoretical side, the weakly interacting BECs provide a fascinating tool for testing theoretical ideas in various fields where experimental control is not possible (such as cosmology and astrophysics, see some comments in the Conclusion). Recently there has been a number of theoretical investigations into topological (and non-topological) defects such as domain walls, vortices, skyrmions and Q-balls in Bose-Einstein condensates [32]–[46]. In some of these works [41]–[44] a nontrivial vortex ring in a two-component BEC is interpreted as the skyrmion, a topologically stable solution originally introduced in particle physics literature [47]. We, however, believe that configurations described in [41]–[44] are not truly topological as the original skyrmions,[1] but semi-topological, owing their stability to a combination of $S^1 \to S^1$ topology and energetics, thus greatly resembling vortons discussed in this paper. The advantage of the analytical approach advocated in the present

[1] We believe that while the topological analysis based on the $S^3 \to S^3$ mapping presented in [41]–[45] is esthetically attractive, it does not reflect the key ingredients essential for the stability analysis. Indeed, the topological skyrmion charge of the $S^3 \to S^3$ map discussed in [41]–[45] is conserved only if the total condensate density does not vanish anywhere. However, there are no topological reasons preventing the total condensate density from localy vanishing, and one has to rely on energetic considerations like the ones presented in the present paper to exclude such mechanism of skyrmion decay.

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work in comparison to the mainly numerical analysis of [41]–[44] is the clear understanding of the structure of the solution for arbitrary large charge, as well as the nature of the stability of the configuration.

This paper is organized as follows. In section 2 we establish the notation and briefly review the properties of conventional U(1) strings. In section 3, we determine the precise conditions under which string core condensation occurs and construct numerical solutions for ϕ and ψ fields in this regime. In section 4, we describe vortons in non-relativistic physics and demonstrate their classical stability.

2. The lagrangian and the conventional U(1) strings

We wish to consider a non-relativistic system with two types of particles such that two different species are very similar to each other and the number of particles of each type is conserved, i.e. the exact symmetry of the system is $\mathrm{U}(1)\times\tilde{\mathrm{U}}(1)$. We also assume that the s wave scattering lengths between like and unlike species are numerically close to each other, such that the relevant symmetry is even larger, as will be explained below. Finally, we assume here that the trapping potentials are sufficiently wide so that in what follows we neglect all the boundary effects. The last simplification is mainly motivated by our interest in effects, which may occur in very large systems (such as the systems, which appear in cosmological and astrophysical context) rather than in finite volume systems where the trapping potential plays an essential role. Thus, throughout this paper, we consider the following lagrangian describing a system of two weakly interacting ultra-cold Bose-gases of same mass:

$$\mathcal{L} = i\hbar\tilde{\Phi}^\dagger\tilde{\partial}_0\tilde{\Phi} - \frac{\hbar^2}{2m}|\tilde{\nabla}\tilde{\Phi}|^2 + \mu_1|\tilde{\psi}|^2 + \mu_2|\tilde{\phi}|^2 - \frac{1}{2}g_{11}|\tilde{\psi}|^4 - \frac{1}{2}g_{22}|\tilde{\phi}|^4 - g_{12}|\tilde{\psi}|^2|\tilde{\phi}|^2 \qquad (2.1)$$

where $\tilde{\Phi} = (\tilde{\psi}, \tilde{\phi})$ is a doublet of complex scalar fields.[2] The chemical potentials of particles $(\tilde{\psi}, \tilde{\phi})$ are (μ_1, μ_2) and coupling constants g_{ij} are proportional to scattering lengths a_{ij} via: $g_{ij} = \frac{4\pi\hbar^2}{m}a_{ij}$.

If $(\tilde{\psi}, \tilde{\phi})$ denote two hyperfine states of ^{87}Rb, the scattering lengths a_{11}, a_{12}, a_{22} differ by only about 3%. We neglect this difference in what follows, and assume $a_{ij} = a$, $g_{ij} = g$, such that the lagrangian (2.1) can be written in a U(2) notation as follows,

$$\mathcal{L} = i\hbar\tilde{\Phi}^\dagger\tilde{\partial}_0\tilde{\Phi} - \frac{\hbar^2}{2m}|\tilde{\nabla}\tilde{\Phi}|^2 - \frac{g}{2}\left(|\tilde{\Phi}|^2 - \frac{\mu_1}{g}\right)^2 + (\mu_2 - \mu_1)|\tilde{\phi}|^2 \, . \qquad (2.2)$$

In the present work we assume $\mu_1 > \mu_2$, and hence the $\tilde{\psi}$ field will condense, while the $\tilde{\phi}$ field will remain uncondensed in the ground state.

It proves to be very convenient to introduce a dimensionless notation for fields, as well as for spacial (and time) coordinates. In the dimensionless notation all distances are measured in units of the correlation (healing) length $\xi = (\hbar^2/2m\mu_1)^{1/2}$, all frequencies are

[2]The original dimensionfull fields and space-time variables appearing in (2.1) carry a sign "tilde". Throughout this paper we will be mostly using a dimensionless notation for the fields and space-time variables, which will be introduced below; these will carry no sign "tilde".

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measured in units $\mu_1/\hbar$, while the absolute values of the fields $|\tilde{\phi}|^2, |\tilde{\psi}|^2$ are measured in units of particle density $n = \mu_1/g$, see appendix for precise correspondence. Throughout this paper, we use the sign "tilde" to distinguish dimensionless (without tilde) and original, dimensionfull (with tilde) variables. Although we will be primarily using the dimensionless notation, we will sometimes restore all physical units in key formulas; we hope this will not confuse the reader. With all these remarks in mind, our starting point is the following dimensionless lagrangian,

$$\mathcal{L} = i\Phi^\dagger \partial_0 \Phi - \mathcal{H}\,, \qquad \Phi = (\psi, \phi)\,. \tag{2.3}$$

Here the hamiltonian density $\mathcal{H}(\psi, \phi)$ has the form,

$$\mathcal{H} = |\nabla\Phi|^2 + \frac{1}{2}(|\Phi|^2 - 1)^2 + \delta m^2 |\phi|^2\,, \tag{2.4}$$

where all derivatives are with respect to the dimensionless coordinates introduced in the appendix. This lagrangian is exactly equivalent to eq. (2.2). It now is apparent that the only relevant parameter entering the hamiltonian is the difference in chemical potentials of ψ and ϕ fields, which is parameterized by a dimensionless factor $\delta m^2 \equiv (\mu_1 - \mu_2)/\mu_1$. In expression (2.3), Φ is a doublet of complex scalar fields, and the lagrangian possesses an exact $\mathrm{U}(1) \times \tilde{\mathrm{U}}(1)$ symmetry with respect to independent phase rotations of the fields,

$$\mathrm{U}(1): \quad \psi \to e^{i\alpha_1}\psi\,, \qquad \tilde{\mathrm{U}}(1): \quad \phi \to e^{i\alpha_2}\phi\,.$$

The corresponding conservation laws are those of the number of particles of each specie, $N_1 = \int d^3x \psi^\dagger \psi$ and $N_2 = \int d^3x \phi^\dagger \phi$. Moreover, the lagrangian (2.3) possesses an approximate $\mathrm{U}(2)$ symmetry, which is explicitly broken by the $\delta m^2|\phi|^2$ term. In the limit $\delta m \to 0$ the lagrangian acquires an exact $\mathrm{U}(2)$ symmetry,

$$\mathrm{U}(2): \quad \Phi \to e^{i\alpha} e^{i\frac{\sigma^a}{2}\theta^a}\Phi$$

with σ^a being the Pauli $\sigma-$ matrices. All of these symmetries are global, although local U(1) symmetries could also be considered without changing the results of this work qualitatively. Without loss of generality, throughout this paper, we consider the case $\delta m^2 > 0$. Then by minimizing the potential energy, we find that the vacuum expectation value for ψ field is non-zero, which we choose to be $\langle\psi\rangle = 1$, while $\langle\phi\rangle = 0$. Thus, the U(1) symmetry is spontaneously broken, while the $\tilde{\mathrm{U}}(1)$ symmetry remains an exact symmetry. Hence the vacuum manifold of the system is a circle S^1 with points characterized by the phase of the field ψ. As we know, spontaneous breaking of a U(1) symmetry leads to formation of topological defects known as strings. Therefore, one should expect the existence of these objects for arbitrary non-zero values of parameter δm^2.

Let us briefly review here the elementary properties of U(1) strings, which will be used later in the text. Until further notice, we will consider all the fields to be uniform in the 3rd (string) direction and, thus, constrain our attention to a plane crossing the string at a right angle. A string is a time-independent configuration, which exhibits a non-trivial mapping of the plane boundary S^1 into the vacuum manifold. In particular, for the U(1) string,

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as one traverses the circle out at infinity in real space, the phase of the field ψ makes an integer number of windings l in the vacuum manifold S^1. Such configurations are absolutely stable since it is impossible to continuously deform configurations with different values of topological charge l into each other without leaving the vacuum manifold. Observe that if the U(2) symmetry between ψ and ϕ fields was exact (i.e. $\delta m^2 = 0$) then the vacuum manifold would have been given by $|\Phi| = 1$, rather than $|\psi| = 1$. As is well known, there are no topologically stable strings possible in this case as there is always a direction in the configurational space through which any windings of the ψ field can be continuously removed. An important consequence of the above formal fact is that in two component BECs the only reason for the existence of vortices is an explicit violation of the U(2) symmetry. Without this violation, the strings would immediately decay. Thus, we expect the typical characteristics of strings such as the core size to be ultimately related to the magnitude of violation of the U(2) symmetry, see the detailed discussion in the next section.

Let us now construct explicit variational string configurations, which will be used later in the text. Formally, we wish to find a solution to the following equations of motion,

$$\nabla^2\psi(x) = (|\psi(x)|^2 - 1)\psi(x)\,, \qquad \phi = 0\,. \tag{2.5}$$

such that the ψ field satisfies the string boundary conditions, i.e. it carries a winding number l. Here we have assumed that in the lowest energy state possessing the string boundary conditions, the ϕ field nowhere leaves its vacuum expectation value (we will see in the next section that this need not be the case). The standard cylindrically symmetric ansatz corresponding to the string configuration is,

$$\psi(x) = f(r)e^{il\varphi}\,, \qquad f(r\to 0)\to 0\,, \qquad f(r\to\infty)\to 1 \tag{2.6}$$

where (r,φ) are the standard polar coordinates, and $f(r)$ is real. We consider only strings with unit winding number $l = 1$, as strings with higher winding number are unstable with respect to decay into $l = 1$ strings [3].

A convenient variational solution to a vortex has been constructed in [17]. This uses a variational ansatz,

$$f(r) = 1 - e^{-\beta r} \tag{2.7}$$

where β is a free variational parameter, which is determined by minimizing the string tension (energy per unit vortex length) μ:

$$\mu = \int d^2x\mathcal{H} = \int d^2x\left(|\nabla\psi|^2 + \frac{1}{2}(|\psi|^2 - 1)^2\right). \tag{2.8}$$

This minimization yields:

$$\tilde\beta = \frac{\sqrt{89}}{12\xi}\,, \qquad \tilde\mu = \frac{\hbar^2}{2m}n\cdot 2\pi\left(\log(\tilde\beta\tilde\Lambda) + \frac{3}{4}\right) \tag{2.9}$$

where we restored the physical dimensional units. Thus, we notice that in the absence of the second condensate ($\phi = 0$), the radius of the string core is of order $1/\tilde\beta \approx \xi$ - the typical correlation length in the theory.

3. Condensation in the string core

As we know from the previous section, if the parameter δm^2 is zero, the lagrangian is invariant under the symmetry group $\mathrm{SU}(2) \times \mathrm{U}(1) \to \mathrm{U}(1)$ (broken down to U(1)). From topological arguments we know that such a lagrangian does not allow for existence of vortices since the vacuum manifold is a 3-sphere and, therefore, does not have non-contractible loops. In the opposite limit, when δm^2 is relatively large, the residual symmetry group is $\mathrm{U}(1) \times \tilde{\mathrm{U}}(1) \to \tilde{\mathrm{U}}(1)$ and the vacuum manifold is a circle S^1, leading to formation of stable vortex solutions. From these two limiting cases, it is clear that there should be some intermediate region that somehow interpolates (as a function of δm^2) between the two cases. Here is a possible scenario, which as we shall see in a moment turns out to be correct. For large values of δm^2 nothing interesting happens: conventional vortices of ψ field do exist, the symmetry is restored in the string core, the core size is fixed by the standard correlation length ξ and the ϕ field vanishes everywhere in space. However, when δm^2 starts to decrease, at some finite magnitude of δm^2, an instability arises through the condensation of ϕ field inside the vortex core. As the magnitude of δm^2 further decreases, the strength of the ϕ condensate in the core grows and the size of the core, where both ϕ and ψ are far away from their vacuum expectation values, becomes larger and larger. Finally, at $\delta m^2 = 0$ the core of the string fills the entire space, in which case the meaning of the string is completely lost, and we are left with the situation when the U(2) symmetry of the lagrangian is exact: no stable strings are possible as we know from topological arguments.

Actually, it is of little surprise that there is a tendency for the ϕ field to condense in the vortex core. Indeed, if $0 < \delta m^2 < 1$, the potential energy of the system has a global minimum at $\psi \neq 0, \phi = 0$, and a local minimum at $\psi = 0, \phi \neq 0$. Thus, the vacuum structure of the theory is given by condensation of ψ rather than ϕ. Yet recall that in the core of the vortex ψ must vanish, and thus the system is driven in the core towards the local minimum where the ϕ field condenses. However, the energetic benefit of condensing ϕ in the core competes with the tensional energy needed for ϕ to relax back to its 0 vacuum expectation value at infinity. As was hinted in the previous section, the question of whether ϕ indeed condenses in the core is determined by the magnitude of U(2) symmetry breaking parameter δm^2.

After the above presentation of some intuitive arguments, we proceed with the quantitative discussion. Let us take the conventional string configuration described in the previous section by $\Phi_{string} = (\psi(x) = f(r)e^{i\varphi}, \phi = 0)$ and formulate the question of whether this configuration is stable with respect to any small fluctuations. We must stress that for $\delta m^2 > 0$ topology guarantees that small fluctuations cannot destroy the string, in the sense of removing the windings of ψ field. However, topology makes absolutely no predictions regarding the internal structure of the string - in particular, a priori, there is no reason for $\phi \equiv 0$ to be a stable solution to equations of motion in the background of the ψ string.

The problem of determining whether an instability with respect to ϕ condensation occurs has been solved previously [13]. This is done by expanding the energy of the system 2.4 in the background of a ψ string to second order in ϕ and seeing whether any negative energy modes exist. The result is that there is a critical value of δm^2, $0 < \delta m_c^2 < 1$ such that for $\delta m^2 < \delta m_c^2$ a negative energy mode appears and the string obtains a non-trivial ϕ core. This implies that the $\tilde{\mathrm{U}}(1)$ symmetry gets "spontaneously broken" in the core. The above formal fact has very important consequences such as the existence of a Goldstone mode capable of propagating along the core of the string. On the other hand, if $\delta m^2 > \delta m_c^2$ all modes increase the energy of the system, hence the string solution discussed in the previous section with $\phi = 0$ is absolutely stable and both U(1) symmetries are restored in the core center. The precise value of δm_c^2 has been found numerically in [13] to be $\delta m_c^2 \approx 0.25$.

We would like to obtain explicit solutions as a function of δm^2 for the radial profiles $\phi(r), \psi(r)$ since they will tell us how the vortex evolves as we approach the topologically unstable limit $\delta m^2 \to 0$ by varying the external parameters μ_1, μ_2. It will turn out that by answering this apparently theoretical question we will find a way to test the system for core condensation experimentally by probing the internal structure of the vortex. In the previous work [13] the profile $\phi(r)$ has been determined for δm^2 close to δm_c^2, when ϕ condensation is weak and "self interactions" of the ϕ field as well as back reaction of the ϕ field onto the string profile $\psi(r)$ can be ignored. However, when δm^2 becomes close to 0, the magnitudes of ϕ and ψ fields in the core become comparable (as expected from symmetry considerations) and the ϕ field can no longer be treated as a weak perturbation to the background ψ field.

In this paper we adopt the numerical approach to treat the above problem. The full Euler-Lagrange equations for fields (ψ, ϕ) are:

$$\begin{aligned} \nabla^2\psi - (|\phi|^2 + |\psi|^2 - 1)\psi &= 0 \\ \nabla^2\phi - (|\phi|^2 + |\psi|^2 - 1)\phi - \delta m^2\phi &= 0 \,. \end{aligned} \tag{3.1}$$

Assuming a cylindrically symmetric solution, $\psi(x) = f(r)e^{i\varphi}$, $\phi(x) = \phi(r)$, equations (3.1) reduce to:

$$f'' + \frac{1}{r}f' - \frac{1}{r^2}f - (\phi^2 + f^2 - 1)f = 0 \tag{3.2}$$

$$\phi'' + \frac{1}{r}\phi' - (\phi^2 + f^2 - 1 + \delta m^2)\phi = 0 \tag{3.3}$$

The boundary conditions on f and ϕ are $f(0) = 0$, $f(\infty) = 1$, $\phi'(0) = 0$, $\phi(\infty) = 0$. The condition $\phi'(0) = 0$ is necessary for $\nabla^2\phi$ at the origin to be finite. Before we demonstrate the explicit numerical solutions to equations (3.2), (3.3), it is instructive to calculate the asymptotic behavior of $\phi(r)$ at large r, which provides us with a qualitative picture of the core size. In this case, $1 - f(r)^2$ as well as $\phi(r)^2$ are small so that eq. (3.3) can be linearized to obtain:

$$\phi'' + \frac{1}{r}\phi' - \delta m^2\phi = 0 \,. \tag{3.4}$$

Solution of eq. (3.4) is known to be:

$$\phi(r) \sim \mathrm{K}_0(\delta m\, r) \sim e^{-\delta m r} \,, \tag{3.5}$$

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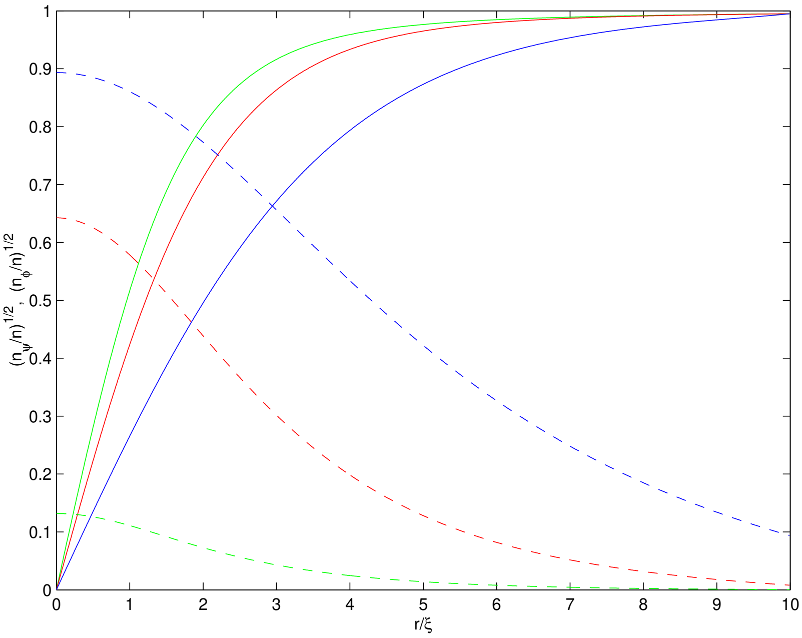

Figure 1: Vortex profile function $|\psi(r)|$ (solid) and core condensate profile function $\phi(r)$ (dashed) for different values of asymmetry parameter $\delta m^2 = (\mu_1 - \mu_2)/\mu_1$. Green, red and blue curves correspond to $\delta m^2 = 0.25, 0.15$ and 0.05 respectively. The quantities plotted are square roots of particle densities (in units of bulk density n) as a function of distance from the vortex core center r (in units of correlation length ξ).

where K_0 is the modified Bessel function. The behavior of solution (3.5) at large r suggests that the distance scale (in physical units), over which ϕ condenses is of order $\xi/\delta m$ rather than the correlation length ξ. Hence we expect that as $\delta m^2 \to 0$ the width of ϕ condensate in the core grows. But ϕ and ψ particles always try to minimize their common territory. Thus, we expect that the region of the string where $|\psi| \neq 1$ also has the length scale $\xi/\delta m$ rather than the typical correlation length ξ. As we will demonstrate below, our numerical results confirm this intuitive reasoning. We conclude that as $\delta m^2 \to 0$, the core of the string expands and eventually fills the whole space. So the string disappears in the limit $\delta m^2 = 0$ in accordance with topological arguments.

Let us now explicitly test the above predictions. We solve numerically the system (3.2), (3.3) for different values of δm^2. Figure 1 displays the numerical solutions $f(r)$ and $\phi(r)$ for $\delta m^2 = 0.25, 0.15$ and 0.05. Essentially, figure 1 demonstrates how the densities of condensed particles $\sqrt{n_\psi(r)/n} = |\psi(r)|$ and $\sqrt{n_\phi(r)/n} = |\phi(r)|$ vary near the vortex core with distance measured in units of the correlation length ξ. From figure 1 it is evident that as δm^2 decreases, the magnitude and width of ϕ condensation in the core increases considerably, and the core of the ψ vortex expands. Thus, the numerical calculations support our intuitive explanation given above. The expansion of the vortex core due to core condensation of the second component may have very important practical consequences. Typically, the core size in BEC experiments cannot be directly observed because $\xi \leq 1\mu m$, which usually is below the resolution limit. We speculate at this point that one can adjust the asymmetry of the system in such a way that the core condensation occurs. Further decreasing the asymmetry by adjusting μ_1, μ_2, one can make the core large enough (larger than the resolution limit) to allow direct detailed experimental measurements of vortex core properties, such as the core radius and the density of the second component ϕ filling the core.

Summarizing the results of this section, we have demonstrated that in the case when the U(2) symmetry of lagrangian (2.3) is broken only weakly, the ψ strings might obtain non-trivial cores. Inside of these cores, the ϕ field is condensed and the $\tilde{\mathrm{U}}(1)$ symmetry is spontaneously broken. We have found that the necessary and sufficient condition for core condensation is $0 < \delta m^2 \lesssim 0.25$, which implies that the effect takes place even if the difference in chemical potentials of the two components is as large as 25%. The described effect becomes much more pronounced if the asymmetry parameter is small. When the asymmetry $\delta m^2 \to 0$, the string core expands and eventually fills the whole space, destroying the topological defect.

4. Vortons

4.1 Stable string loops

The most important result of the previous section is that under certain conditions a ϕ condensate will be trapped inside the ψ string. In this case, according to Goldstone's theorem, a new degree of freedom corresponding to the phase of the ϕ field becomes a massless Goldstone boson, which can travel along the core of the string with arbitrarily low energy cost.

Now, we follow Witten [5], who has constructed his superconducting string by making a loop out of the vortex with a second condensate in the core. Therefore, let us consider bending the ψ string with a ϕ core into a loop of large radius R. In order to ignore curvature effects, we take R to be much greater than the width of the string core. Such a configuration is certainly unstable from the topological point of view since the change of argument of ψ along a large circle surrounding the loop is always 0 (so the loop belongs to the trivial topological class). From another point of view, the energy of the loop is $E = 2\pi\mu R$, where μ is the string tension computed in section 2. Thus, the energy E is minimized when $R = 0$. The fact that a simple string loop shrinks with time in the relativistic setting has been confirmed by numerical simulations [24], and the same behavior should be expected in the non-relativistic case provided the loop does not move and the Magnus force does not equilibrate the string tension.

Nevertheless, as was originally proposed for the case of cosmic strings [16]–[23], a string loop can be stabilized by a persistent current carried by the phase of the core condensate ϕ along the loop. Here we discuss this original idea applied to our non-relativistic system. Indeed, suppose that the phase of the ϕ field varies by $2\pi N$, as one traverses the string loop, where N is constrained to be an integer since ϕ is single valued. Then N acts as a conserved number: in order for N to change, the field ϕ has to vanish somewhere on the string core, which is energetically unfavorable by the results of the previous section. N is said to be a semi-topological charge since it is conserved in part due to topological reasons and in part due to energetics. We can explicitly parameterize the field ϕ of this configuration as $\phi(x) = \phi(r)e^{ikz}$, where r is the distance to the core center in the plane perpendicular to the string direction, so that $\phi(r)$ is the same as in the previous section, and z is the spatial variable running along the string core. Then the variation of the phase of ϕ along the string loop is $2\pi kR$, so we can identify, $k = N/R$.

In a non-relativistic system, there is always present another conserved number — the Noether's charge associated with the $\tilde{\mathrm{U}}(1)$ symmetry, which is simply the total number of ϕ particles. This is given by,

$$Q = \int d^3x|\phi|^2 \,. \tag{4.1}$$

As we demonstrated previously, it is unfavorable for ϕ particles to leave the string, so the conserved charge Q will be confined to the string core, being effectively trapped by the dynamics rather than by some external forces.

Notice that the variation of the phase of ϕ along the string influences the energy of the configuration through the term $|\partial_z\phi|^2$ in the energy density. When integrated over 3-dimensional space, this term contributes the following amount into the energy of the system,

$$\Delta E = k^2 \int d^3x|\phi|^2 = \frac{N^2}{R^2}Q. \tag{4.2}$$

Thus, ignoring other contributions for a moment (detailed analysis of this problem will be presented in section 4.3), the energy of the loop becomes,

$$E \approx 2\pi\mu R + \frac{N^2}{R^2}Q \,. \tag{4.3}$$

Thus, as the charges N and Q are conserved within the configuration, the energy of the vortex loop has a non-trivial minimum with respect to the loop radius at $R_0 = \sqrt[3]{N^2Q/\pi\mu}$. Therefore, it is expected that the loop will stop shrinking at the above radius and a classically stable configuration known as vorton will form. The stability is purely mechanical in origin and can be thought to be due to the angular momentum $M = QN$ carried by the vorton (see section 4.4).

The formula (4.3) is quite important, so we want to restore all dimensional factors to represent the final result as follows,

$$\tilde{E}(\tilde{R}_0) = 3\pi\tilde{\mu}\tilde{R}_0 \,, \qquad \tilde{R}_0 = \sqrt[3]{\frac{\hbar^2}{2m}\frac{N^2\tilde{Q}}{\pi\tilde{\mu}}} \tag{4.4}$$

where $\tilde{\mu}$ is determined by expression (2.8) and $\tilde{Q}$ is the physical number of trapped ϕ particles. The most striking feature of the vorton is that its size could, in principle, be arbitrarily large. Practically, the size is mainly limited by the boundary conditions specific to a given experiment and by our ability to produce a configuration with a large winding number N and charge Q during the formation period.

It is important to note that the vortons described here have nothing in common with vortex loops (observed in the superfluid ^{4}He, see [4]) stabilized by Magnus force, which is the interaction of a moving string with its environment. String rings discussed in the present work can be stable even if they do not move as a whole object. The non-relativistic

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configuration bearing some similarity to the one described here is the twisted vortex ring (vorton) discussed in the superfluid ^{3}He-B system at low temperature and pressure, see [6] and references therein. Although it was originally hoped that vortons can become stable in this system, numerical estimates have shown [6] that it is unlikely to happen, and such rings have never been observed in ^{3}He-B. Actually, vortex ring formation and dynamics in BECs were discussed earlier [32]–[36]. However, in most cases the analysis of ring stabilization (as opposed to formation) was done for a one-component BEC, where the stability (or meta-stability) is due to the combination of ring motion and details of trapping potential. In contrast, our main objective is the analysis of vortons, which can be stable in an infinitely large volume without moving; this is the kind of system, which might be of interest for cosmology and astrophysics. A close analogue of a vorton is the full 3D skyrmion, which was recently discussed in the context of 2-component BECs [41]–[45]. Similarly to a vorton, the skyrmion can be represented as a vortex ring in one component with the second component carrying an integer quanta of circulation around the ring. Numerical simulations [41]–[44] of the skyrmion solution in BECs resemble very much the analytical construction of vortons discussed here. We suspect that the stability of configurations considered in [42, 43, 44] is due to a combination of energetics and topology, rather than to topology alone (see footnote 1). We must stress that these simulations were performed for small values of topological charge only, thus, producing point-like objects of small size. On the other hand, the size of vortons is determined by quantum numbers Q and N and could, in principle, be arbitrarily large as explained below.

4.2 Relativistic springs and vortons

The detailed construction of vortons in non-relativistic physics will be discussed in the next subsection. However, we believe that a historical aside will be instructive here: we now make a detor to briefly describe stable vortex loops, which were first considered in the context of cosmic strings [16, 17] (for a full review of the subject see [23]). Later on, the ideas developed in the above works have been applied to systems with high baryon density [14] and to high T_c superconductors [9]. The lessons learned from these papers prove to be very useful for our discussion of vortons in non-relativistic systems.

In this subsection, we consider the phenomenon of string core condensation and vorton stability in the relativistic system defined by the following lagrangian,

$$\mathcal{L} = |\partial_0 \Phi|^2 - \mathcal{H}\,, \tag{4.5}$$

where $\mathcal{H}$ is the same as before (2.4). Here we use the relativistic notation with $\hbar = c = 1$. The Noether's charge associated with the $\tilde{\mathrm{U}}(1)$ symmetry in the relativistic case is:

$$Q = \frac{i}{2}\int d^3x(\partial_0\phi\,\phi^* - \partial_0\phi^*\,\phi)\,. \tag{4.6}$$

It is important to note that in the relativistic setting, the ϕ condensate in the core of the string does not necessarily carry a $\tilde{\mathrm{U}}(1)$ Noether's charge. In particular, all static solutions satisfy $Q = 0$.

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The first attempts to stabilize the string loops were based on the introduction of a winding number density $k = N/R$ of the core condensate ϕ, exactly as described in the previous section. These relativistic objects, dubbed springs, were characterized by $N \neq 0$ and $Q = 0$. It was soon realized, however, that such objects, in general, are not stable: persistent current related to the winding number N cannot stabilize the string loops against shrinking. Fortunately, a related mechanism which makes string rings stable with respect to classical decay was found by Davis and Shellard [18]–[21]. The idea is to provide the configuration with a nonzero charge (4.6) trapped in the string core along with the winding number N; such a configuration carries an angular momentum QN, hence its name — "vorton". According to (4.6), the presence of charge Q automatically implies time dependence of the ϕ field, which can be parameterized as $\phi(\vec{x},t) = e^{i(kz-\omega t)}\phi(r)$, where $\phi(r)$ is the profile function similar to one discussed previously. Naively, one could think that time dependence of a classical solution brings in additional energy into the system, which usually further undermines stability. However, as one can show, the stability is enforced by conservation of charge Q. In a sense, the time-dependent configuration becomes the lowest energy state in the sector with given quantum numbers: non-zero charge Q and winding number N. A similar time-dependent ansatz for a different problem was discussed by Coleman in [48] where he introduced the so-called Q-balls: macroscopically large stable objects with time dependent fields (for a recent discussion of Q-balls in BECs, see [46]). It is currently believed that similarly to Q-balls, relativistic vortons are classically stable objects. The recent, full scale numerical simulations [22] support this statement. In the non-relativistic models to which our paper is devoted, a Noether's charge Q is always present when the field ϕ is condensed in the core, so the vorton stabilization mechanism is very similar to the one suggested by Davis and Shellard.

4.3 Non-relativistic vortons

Let us now demonstrate in detail how the string loop shrinks and how the vorton stability is achieved in a non-relativistic system. We are interested in configurations characterized by two quantum numbers,

$$N = \frac{1}{2\pi}\int_C d\arg\phi(x)\,, \qquad Q = \int d^3x|\phi|^2 \tag{4.7}$$

where C is the path along the core of the string loop. N is the topological charge, and Q is a $\tilde{\mathrm{U}}(1)$ Noether's charge, which is just the total number of type ϕ particles. Fixing the charges Q, N, and also for now fixing the loop radius R, we wish to determine the corresponding fields. We can adopt an ansatz for the field ϕ motivated previously,

$$\phi(x,t) = e^{i(kz-\omega t)}\phi(r) \tag{4.8}$$

where $\phi(r)$ is the radial profile of the core condensate in the plane perpendicular to the string direction. As already explained, k acts as a winding number density, and $k = N/R$. As we will see shortly, ω can be identified with the Lagrange multiplier for conservation of charge Q within the configuration under discussion. At this point, ω is still a free parameter; it will be later eliminated by solving the equations of motion using the ansatz (4.8) and substituting the solution into the constraint for conservation of charge Q (4.7). A more precise physical meaning of ω will be given later in the text.

Substituting the ansatz (4.8) into the action corresponding to lagrangian (2.3), we obtain:

$$S[\Phi] = \int dt \left(\omega \int d^3x |\phi(x)|^2 - \int d^3x \mathcal{H}[\Phi] \right) \tag{4.9}$$

where $\mathcal{H}$ is the full hamiltonian density of the system defined by eq. (2.4). The Lagrange multiplier nature of ω is now evident - the energy of the system is minimized subject to constraint $\int d^3x|\phi(x)|^2 = Q$. In terms of the core condensate profile $\phi(r)$ (see eq. 4.8), the above constraint requires,

$$\Sigma_2 \equiv \int d^2x |\phi(r)|^2 = \frac{Q}{2\pi R} \, . \tag{4.10}$$

The physical meaning of the parameter Σ_2 introduced above is quite obvious: it is the number of ϕ particles trapped in the core per unit length of the string. In physical units,

$$\widetilde{\Sigma}_2 = \frac{\tilde{Q}}{2\pi \tilde{R}} = \int d^2\tilde{x} |\tilde{\phi}|^2 = n\xi^2 \Sigma_2$$

where $\tilde{Q}$ is the physical total number of trapped ϕ particles and Σ_2 is the dimensionless number in eq. (4.10).

We want to construct an effective lagrangian describing the "dynamics" of the field $\phi(r)$ on the two-dimensional xy plane perpendicular to the string axis, which points in the z direction, see (4.8). The effective lagrangian is obtained by substituting the ansatz (4.8) into the lagrangian (2.3),

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= -|\nabla \Phi|^2 - \frac{1}{2}(|\Phi|^2 - 1)^2 - \widehat{\delta m}^2 |\phi|^2 \, , \\ \widehat{\delta m}^2 &\equiv (\delta m^2 + k^2 - \omega) \, , \end{aligned} \tag{4.11}$$

where the gradient is with respect to two coordinates (x, y) perpendicular to the axis of the string. Hence the dynamics of the system on the xy plane are determined by the new effective parameter $\widehat{\delta m}^2$ defined as follows:

$$\widehat{\delta m}^2 = \frac{\left((\mu_1 - \mu_2) + \frac{\hbar^2}{2m} \tilde{k}^2 - \hbar \tilde{\omega} \right)}{\mu_1} \, , \tag{4.12}$$

where we restored in this expression all dimensional factors to stress its physical significance. This parameter plays the same role as the U(2) asymmetry δm^2 in section 3: the equations of motion for profile functions $\psi(r)$ and $\phi(r)$ imposed by the lagrangian (4.11) are the same as eq. (3.1), but with δm^2 replaced by $\widehat{\delta m}^2$:

$$\nabla^2 \psi - (|\phi|^2 + |\psi|^2 - 1)\psi = 0 \tag{4.13}$$

$$\nabla^2 \phi - (|\phi|^2 + |\psi|^2 - 1)\phi - \widehat{\delta m}^2 \phi = 0 \tag{4.14}$$

where all derivatives are with respect to coordinates (x, y) in the plane perpendicular to string axis.

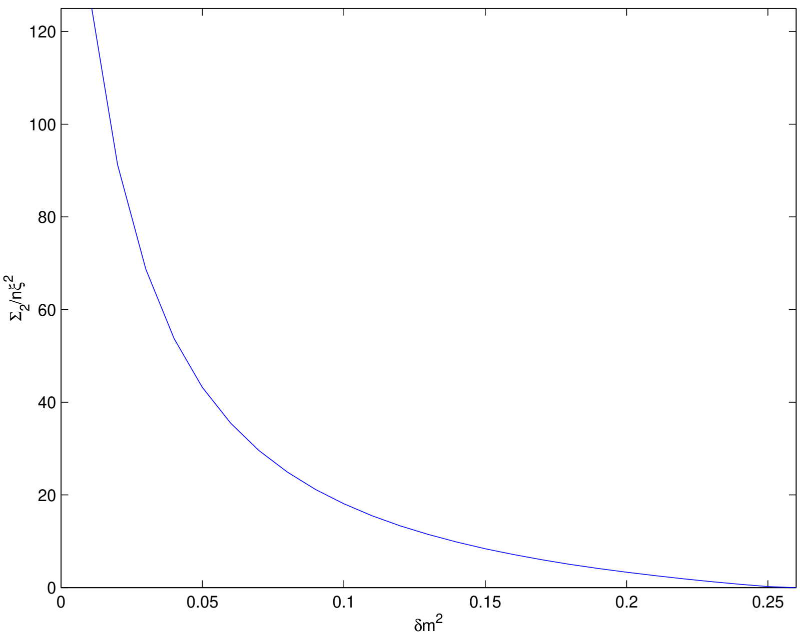

Figure 2: Number of ϕ particles trapped in the vortex core per unit vortex length, Σ_2 (measured in units of $n\xi^2$) as a function of effective asymmetry parameter $\widehat{\delta m}^2 = (\mu_1 - \mu_2 + \frac{\hbar^2 k^2}{2m} - \hbar\omega)/\mu_1$.

Following the procedure of section 3, for each value of $\widehat{\delta m}^2$, we numerically solve the equations of motion (4.13), (4.14) subject to string boundary conditions, to determine the profile functions $\psi(r)$ and $\phi(r)$. We then use the solution $\phi(r)$ to find Σ_2 (see eq. (4.10)) as a function of $\widehat{\delta m}^2$. Figure 2 shows how the ϕ particle density per unit vortex length, $\Sigma_2 = \widetilde{\Sigma}_2/(n\xi^2)$, depends on the effective asymmetry parameter $\widehat{\delta m}^2$. As one can see from this plot, the magnitude of Σ_2 increases drastically when $\widehat{\delta m}^2$ decreases. This is due to the combination of two effects. First, the magnitude of the ϕ field in the core increases when $\widehat{\delta m}^2$ decreases, as shown in figure 1. Second, the core size itself increases with decrease in $\widehat{\delta m}^2$. Also, as figure 2 demonstrates, Σ_2 vanishes when $\widehat{\delta m}^2 > \delta m_c^2 \approx 0.25$, since as was demonstrated in section 3, the condensation of the ϕ field in the core becomes unfavorable for large values of U(2) asymmetry parameter.

Now we wish to understand the physical meaning of the parameter ω, which was originally introduced in a formal way as a Lagrange multiplier. We define our configuration by fixing the quantum numbers Q, N and the loop radius R. To possess these quantum numbers the fields must satisfy the constraints (4.7), which fix the values of $k = N/R$ and $\Sigma_2 = Q/2\pi R$. Yet, as we have shown above, there is a one-to-one correspondence between Σ_2 and the effective symmetry breaking parameter $\widehat{\delta m^2}$ (see figure 2). Thus, the value of $\widehat{\delta m^2}$ is also fixed by the external parameters (Q, N, R). This, in turn, implies that ω is fixed via eq. (4.11) and hence ω is completely determined by initial assignments of (Q, N, R). Now trading external parameters (Q, N, R) for (Q, N, k), we speculate that $\omega(k)$, implicitly determined by eq. (4.11), is just a complicated dispersion relation describing how the Goldstone mode propagates along the string loop in the presence of charge Q and winding number N.

Our next task is the analysis of energetics of the vortex loop, with the goal of finding how the ring energy depends on its radius R. In what follows we apply the technique developed in [23] to the analysis of our non-relativistic system. Starting from the lagrangian (2.3), we switch to a hamiltonian formalism and represent the total energy of the configuration in the following way,

$$\begin{aligned} E &= 2\pi\mu R + \int d^3x \left(|\nabla\phi|^2 + (|\psi|^2 - 1 + \delta m^2)|\phi|^2 + \frac{1}{2}|\phi|^4 \right) \\ &= 2\pi\mu R + 2\pi R \int d^2x \left[\phi^*(-\nabla^2 + |\psi|^2 - 1 + \delta m^2 + k^2)\phi + \frac{1}{2}|\phi|^4 \right], \end{aligned} \tag{4.15}$$

where $2\pi\mu R$ term is the conventional contribution (2.8) due to the ψ field, and in the last step we have performed an integration by parts. The laplacian in the last equation is taken with respect to the (x, y) variables perpendicular to the string axis. The above expression can be simplified via the Euler-Lagrange equation for the field $\phi(r)$ (4.14). Multiplying (4.14) by the complex conjugate $\phi^*(r)$ and then inserting the result into (4.15), we find:

$$E = 2\pi\mu R + 2\pi R \int d^2x \left(\omega|\phi|^2 - \frac{1}{2}|\phi|^4 \right) = 2\pi\mu R + 2\pi R \left(\omega\Sigma_2 - \frac{1}{2}\Sigma_4 \right) \tag{4.16}$$

where $\Sigma_4 \equiv \int d^2x|\phi|^4$. A similar expression for the relativistic vortons was derived in [23]. As we discussed earlier, ω in this expression can be computed by inverting the constraint (4.10), (4.11) producing a complicated implicit function of k, which is difficult to analyze in general. However, for relatively large ω, such that $\omega > \delta m^2, \widehat{\delta m^2}$ (which are smaller than $\delta m_c^2 \approx 0.25$) one can use eq. (4.11) to approximate $\omega \approx k^2 = N^2/R^2$. The physical meaning of this simplification is that the typical energy scale of the ϕ field excitations along the string is larger than the typical scale of excitations perpendicular to the string. To further simplify our discussion, we would like to consider the simple case when the core condensation is not very large and, therefore, the back reaction of the ϕ field onto the ψ vortex can be neglected, so that one can use the unperturbed expression (2.9) for the string tension μ (2.8). Also, one can check that in this regime $\Sigma_4 \ll \Sigma_2$, so the last term in equation (4.16) can be dropped, and the total energy of the configuration becomes,

$$E = 2\pi\mu R + \frac{N^2}{R^2}Q\,, \tag{4.17}$$

where we replaced $\omega \to k^2 = N^2/R^2$. It is interesting to note that the equation (4.17) exactly coincides in form with the one (4.3) derived with a number of simplifications mainly for demonstrative purposes. Thus, as was originally pointed out, conservation of charges Q and N guarantees that the loop energy E has a non-trivial minimum with respect to radius R at,

$$R_0 = \sqrt[3]{\frac{N^2Q}{\pi\mu}}\,, \qquad E(R_0) = 3\pi\mu R_0\,. \tag{4.18}$$

The corresponding expressions with restored dimensional parameters were presented earlier in (4.4). As we argued above, we expect that at this radius the string loop will become

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stabilized against further shrinking, forming the so-called "vorton". The winding number density k and the particle number density Σ_2 corresponding to the radius R_0 are:

$$k = \sqrt[3]{\pi\mu} \cdot \sqrt[3]{\frac{N}{Q}}, \qquad \Sigma_2 = \frac{\sqrt[3]{\pi\mu}}{2\pi} \cdot \sqrt[3]{\left(\frac{Q}{N}\right)^2}. \tag{4.19}$$

Due to the importance of these expressions as explicitly measurable quantities, we would like to present them with all dimensional factors included,

$$\tilde{k} = \sqrt[3]{\frac{2m\pi\tilde{\mu}}{\hbar^2}} \cdot \sqrt[3]{\frac{N}{\tilde{Q}}}, \qquad \widetilde{\Sigma}_2 = \frac{1}{2\pi}\sqrt[3]{\frac{2m\pi\tilde{\mu}}{\hbar^2}} \cdot \sqrt[3]{\left(\frac{\tilde{Q}}{N}\right)^2}. \tag{4.20}$$

One immediate observation is that if one neglects a somewhat weak $\ln R$ variation of μ with R, see (2.9), the local parameters such as k and Σ_2 depend solely on the ratio Q/N. Hence, the local parameters are approximately scale-invariant under the transformation $(Q, N) \to (\alpha Q, \alpha N)$, where α is an arbitrary transformation parameter. Conversely, the global characteristics such as the radius R_0 and the energy E are extensive parameters and transform as $(R, E) \to (\alpha R, \alpha E)$. Thus, one should expect the existence of stable vortons of arbitrarily large size. This statement, of course, should be corrected when finite volume effects, such as the trapping potential, are taken into consideration.

One can present the arguments showing that the back-reaction of the ϕ field onto the ψ vortex is negligible for vortons in a large region of (Q, N) parameter space, where, in particular, the parameter ω is large compared to $\delta m^2, \widehat{\delta m^2}$. Thus, both assumptions made in our discussion of energetics of vortex loops with core condensates are justified a-posteriori. One should note that when either of these two assumptions is not valid, we still expect that stable vortons will generally exist. However, the analysis of vorton stability becomes very involved in this case as the string tension μ and the frequency ω become very complicated functions of (Q, N, R), and no analytical control is possible. We have not discussed this case because our main goal was to demonstrate the fact of vorton existence in a large region of the parametrical space rather than a complete analysis of the allowed region of parametrical space where vortons are stable.

4.4 Vortons and angular momentum

In this section we demonstrate that the non-relativistic vorton carries an angular momentum. From lagrangian (2.3), the total angular momentum of the system is:

$$\vec{M} = \int d^3x \Big(\Phi^\dagger \, [\vec{x} \times -i\vec{\nabla}] \Phi \Big). \tag{4.21}$$

When the Φ field forms a vorton, there is no net circulation of ψ particles and the only contribution to the angular momentum comes from the ϕ field condensed in the core. We can re-parameterize the ansatz (4.8) as:

$$\phi(x) = e^{iN\theta}\phi(\rho, z) \tag{4.22}$$

where (ρ, θ, z) are the polar coordinates with respect to the center of the vortex loop. Substituting this into (4.21) gives:

$$\vec{M} = N\hat{z} \int d^3x|\phi|^2 = QN\hat{z}\,, \tag{4.23}$$

which is a very simple relation that can be easily understood on the intuitive level in terms of classical physics.

Therefore, the vorton carries an angular momentum perpendicular to the string loop plane. The magnitude of the angular momentum is proportional to classically conserved charges Q and N . Thus, as originally suggested in the relativistic context by [20], the stability of vortons can be understood in terms of conservation of angular momentum. Let us note that due to the scaling properties of eqs. (4.17), (4.23) it is energetically favorable for two identical vortons, each with an angular momentum M, to merge into one vorton with an angular momentum $2M$, thereby increasing the vorton radius (this statement is subject to the assumption that that the number of ϕ particles trapped in the ring core is conserved in this merge). So we expect that a typical vorton will possess, in general, the largest possible size constrained only by the boundary conditions and/or by the properties of the trapping potential.

5. Conclusion

The main goal of this work is to demonstrate that in a non-relativistic system, such as a two component BEC, with a $\mathrm{U}(1) \times \tilde{\mathrm{U}}(1)$ symmetry and an approximate $\mathrm{U}(2)$ symmetry, stable strings with a non-trivial core may exist. Inside this core the $\tilde{\mathrm{U}}(1)$ symmetry is spontaneously broken allowing a Goldstone boson to travel along the string axis. We have explicitly demonstrated that within our model the condensation inside the core occurs even for relatively large values of $\mathrm{U}(2)$ symmetry breaking parameter, $\delta m^2 \lesssim 0.25$. The effect described becomes much more pronounced when the $\mathrm{U}(2)$ asymmetry becomes even smaller. We should note that the phenomenon when a string core possesses a condensate of a different field/phase is by no means a completely new situation in physics. A similar phenomenon takes place in ^{3}He and in high T_c superconductors where it has been studied experimentally quite extensively. Nevertheless, it would be very interesting to study this very nontrivial phenomenon experimentally in two component BEC systems, with the specific focus on the dependence of the effect on the asymmetry parameter δm^2. In particular, we believe that with decrease in asymmetry, the vortex core will expand above the resolution constraints, therefore, providing an opportunity to experimentally study the core in detail.

In the second part of the paper we applied the idea of $\tilde{\mathrm{U}}(1)$ symmetry breaking inside the core to construct stable string loops known as vortons. These vortons are stabilized by a winding number carried by the core condensate field around the loop as well as a Noether's charge associated with the $\tilde{\mathrm{U}}(1)$ symmetry trapped in the string core. We have seen that the energy of such a string loop will have a non-trivial minimum with respect to the loop radius where the loop will become stabilized against further shrinking. We have also shown that non-relativistic vortons carry an angular momentum, which can be seen as the source of their classical stability.

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It is puzzling that although, in general, the vorton is expected to be a fairly common configuration in quantum field theory, as far as we can tell, it has never been observed in nature. Indeed, at this point, discussions of cosmic strings [5] in cosmology and vortons [14] in high baryon density systems should be considered as no more than beautiful speculations. Some systems proposed earlier to test the existence of vortons are high T_c superconductors [9] and liquid ^{3}He. However, currently no direct observations can be presented to support the existence of vortons in high-T_c superconductors, while in the ^{3}He system, estimates suggest that vortons cannot be stabilized [6, 7]. Thus, the BEC system might provide us with the first true experimental opportunity to prove the existence and stability of vorton configurations that have been the subject of intense theoretical discussions for the past 20 years.

It is tempting to identify the vortex rings observed in the BEC system [31] with vortons, which are the subject of this work. However, such an identification would be premature. Actually, in the original experiment, a two-component system was initially constructed, but the second component was removed before the rings were created. Hence the experimental results were compared by [31] to a simulation of a one component BEC [32], where vortons cannot exist, as there is no second component capable of condensing in the vortex core. If, indeed, the observed rings are due to one field component only then they may be analogous to the vortex rings in ^{4}He [4], where the ring stability is due to the Magnus force. In this case, the rings must move in order to equalize the string tension that causes them to shrink. Ring stability might also be due in part to the artificial trapping potential [35, 36]; this situation is not addressed in our work.

However, granted that some population of the second component remained in the original experiment, the vortex rings observed in [31] could be also real vortons, which are stable due to the condensation of the second field in the core. In this case it would be the first ever observation in nature of such objects. There is a "smoking gun" signature, which distinguishes between conventional vortex rings, similar to the ones observed in ^{4}He, and vortons, which have never been previously observed. This signature is the angular momentum (4.23), which is always present for vortons and absent for conventional vortex rings.[3] We do not know whether the angular momentum of the ring (or equivalently the circulation of the second component in the core of the ring) can be measured in experiments similar to [31]. Another vorton signature is its ability to stay still for a relatively long period of time. This distinguishes vortons from the conventional vortex rings, which shrink and decay in the absence of motion. We do not know whether the rings, which typically move when they are formed, can be stopped to observe their behavior. However, if either of the two experiments described above can be performed, it would be the "smoking gun" evidence demonstrating the existence of vortons stabilized due to core condensation.

Finally, if vortons can, indeed, be formed and studied in BEC experiments, it would provide a unique opportunity to study not only condensed matter physics, atomic physics

[3]The angular momentum of a conventional ring is due to the contribution of the ψ field into M in eq. (4.21), which is identically zero.

and optics, but also astrophysics and cosmology by doing laboratory experiments. In particular, one would finally be able to test mathematical ideas pioneered by Witten, Turner, Kibble, Davis, Shellard and many others. Over the last few years several experiments have been done to test ideas drawn from cosmology (see the review papers [6, 7, 51, 52] for further details). One can further speculate on a possibility to study the formation, evolution and stability of different topological objects (including vortons) relevant for cosmology by doing controlled experiments with BEC systems as discussed in the recent workshop [53].

Acknowledgments

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A. Physical scales

In this appendix, we reduce the lagrangian (2.2) to a dimensionless form (2.3). In order to do this, we introduce scaled coordinates and fields,

$$\tilde{t}=\frac{t}{\omega_0}\,,\quad \tilde{\vec{x}}=\xi\vec{x}\,,\quad \tilde{\Phi}=\sqrt{n}\Phi\,,\quad \omega_0=\frac{\mu_1}{\hbar}\,,\quad \xi=\left(\frac{\hbar^2}{2m\mu_1}\right)^{1/2}\,,\quad n=\frac{\mu_1}{g}\,,\quad \delta m^2=\frac{\mu_1-\mu_2}{\mu_1}\,. \tag{A.1}$$

We remind the reader that the original (dimensionfull) variables carry a "tilde", while the introduced dimensionless variables carry no "tilde". In the above notation, the action $\tilde{S}$ takes the form,

$$\frac{\tilde{S}}{\hbar}=(n\xi^3)\int dtd^3x\left(i\Phi^\dagger\partial_0\Phi-|\nabla\Phi|^2-\frac{1}{2}(|\Phi|^2-1)^2-\delta m^2|\phi|^2\right) \tag{A.2}$$

where all derivatives are with respect to dimensionless coordinates $(t,\vec{x})$. This is exactly (up to a multiplicative factor) the action corresponding to the dimensionless lagrangian (2.3) used throughout this paper. The physical distance corresponding to one unit of dimensionless variable x is just the correlation length ξ, the physical time corresponding to one unit of dimensionless variable t is $1/\omega_0$ and the physical density corresponding to one unit of $|\Phi|^2$ is just the bulk BEC density n. The typical values of parameters in experiments on ^{87}Rb are $\xi\sim 0.1\mu m$, $n\sim 10^{14}cm^{-3}$, $\omega_0\sim 5kHz$.

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